A characterization of p-automatic sequences as columns of linear cellular automata

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http://arxiv.org/pdf/1209.6008v1.pdf

June 5, 2013

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Theorem[Rowland, Y, 2012]

If a sequence of elements in $\mathbb{F}_q$ is p-automatic, then it is a column of a spacetime diagram of a linear cellular automaton with memory over $\mathbb{F}_q$ whose initial conditions are eventually periodic in both directions. Furthermore, our proof is constructive.

Some pictures

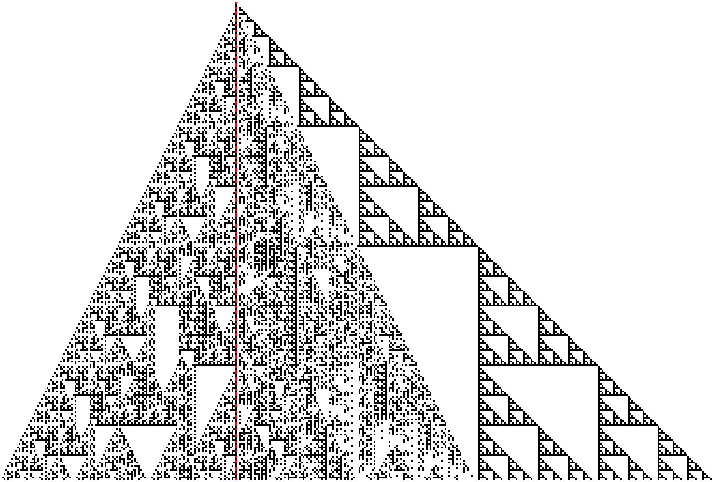

Figure : Spacetime diagram of a linear cellular automaton with memory 12 containing the Thue–Morse sequence as a column.

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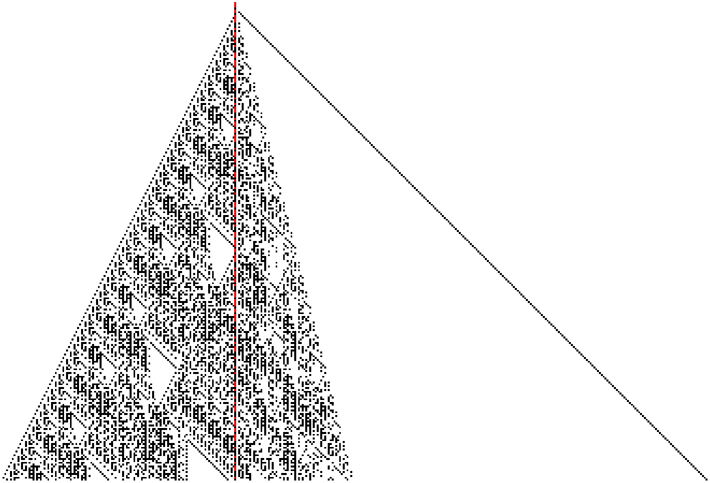

Figure : Spacetime diagram of a linear cellular automaton with memory 20 containing the Rudin-Shapiro sequence as a column.

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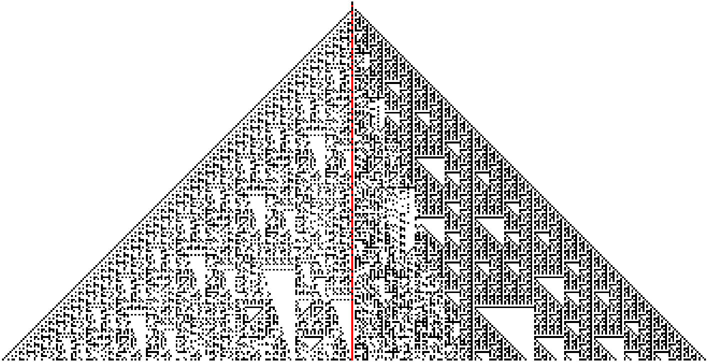

Figure : Spacetime diagram of a linear cellular automaton with memory 27 containing the Baum-Sweet sequence as a column.

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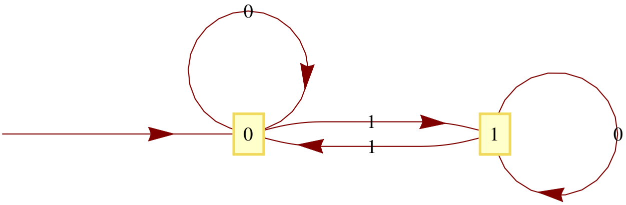

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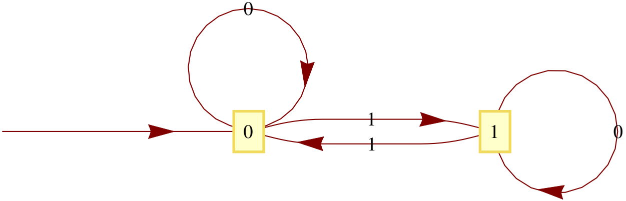

We will work only with p-automatic sequences. By injecting $\mathcal{A}$ into some $\mathbb{F}_q$ with $|\mathcal{A}| \leq q = p^n$, we can assume $\mathcal{A} = \mathbb{F}_q$.

If $n = \sum_{i=0}^{l} a_i k^i$ is the standard base-k representation of n with $0 \leq a_i \leq k-1$ and $a_l \neq 0$, define $(n)_k$ to be the word $a_0\, a_1 \cdots a_l$.

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Definition

A sequence $(u_n)_{n\geq 0}$ of elements in $\mathcal{A}$ is *k-automatic* if there is a DFAO $(\mathcal{S}, \Sigma_k, \delta, s_0, \mathcal{A}, \omega)$ such that $u_n = \omega(\delta(s_0, (n)_k))$ for all $n \geq 0$.

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Example: The Thue–Morse sequence is the 2-automatic sequence $(u_n)_{n \geq 0} = 0, 1, 1, 0, 1, 0, 0, 1, \ldots$ where $u_n = 0$ if the number of occurrences of 1 in the binary representation of n is even and $u_n = 1$ otherwise.

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[Cobham, 1972] A sequence is k-automatic if and only if it is the image, under a letter-to-letter projection, of a fixed point of a length-k substitution.

Cellular automata with memory

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The Curtis–Hedlund–Lyndon theorem also holds for a cellular automaton with memory, so that Φ is a cellular automaton with memory d iff there is a local rule $\phi : (\mathcal{A}^d)^{l+r+1} \to \mathcal{A}$ (l=left radius, r =right radius, $l \geq 0$, $r \geq 0$) such that for all $R \in \mathcal{A}^{d^{\mathbb{Z}}}$ and all $m \in \mathbb{Z}$,

$$(\Phi(R))(m) = \phi(R(m-l), R(m-l+1), \ldots, R(m+r)) \,. \qquad (1)$$

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Example

Rule 90 is an LCA, l=r=1 defined over $\mathbb{F}_2$; its local rule is $(\phi)(a, b, c) = a + c$.

Definition

If $\Phi : (\mathcal{A}^d)^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a cellular automaton with memory d, then a *spacetime diagram* for Φ with initial conditions $R_0, \ldots, R_{d-1}$ is the sequence $(R_n)_{n \geq 0}$ where we inductively define $R_n := \Phi(R_{n-d}, \ldots, R_{n-1})$ for $n \geq d$.

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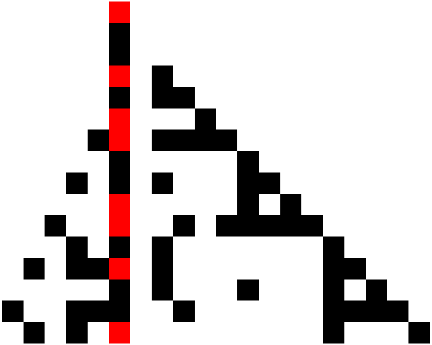

Figure : Spacetime diagram of a linear cellular automaton with memory 12 containing the Thue–Morse sequence as a column.

Corollaries

Theorem[Rowland, Y, 2012]

If a sequence of elements in $\mathbb{F}_q$ is p-automatic then it is a column of a spacetime diagram of a linear cellular automaton with memory over $\mathbb{F}_q$ whose initial conditions are eventually periodic in both directions. Furthermore, the proof is constructive.

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Corollary 1
If $(u_n)_{n\geq 0}$ is a p-automatic sequence, then the sequence $(u_n)_{n\geq 0}$ is the letter-to-letter projection of a sequence $(v_n)_{n\geq 0}$ which occurs as a column of a linear cellular automaton (without memory) whose initial condition is eventually periodic in both directions.

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If $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$, define $X_{\mathbf{u}} := \overline{\{\sigma^n(\mathbf{u}) : n \in \mathbb{N}\}}$. The dynamical system $(X_{\mathbf{u}}, \sigma)$ is called the (one-sided) subshift associated with $\mathbf{u}$.

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Corollary 2
Let $\mathbf{u}$ be p-automatic. Then $(X_{\mathbf{u}}, \sigma)$ is a factor of a subsystem of some linear cellular automaton $((\mathbb{F}_q^d)^{\mathbb{Z}}, \Phi)$.

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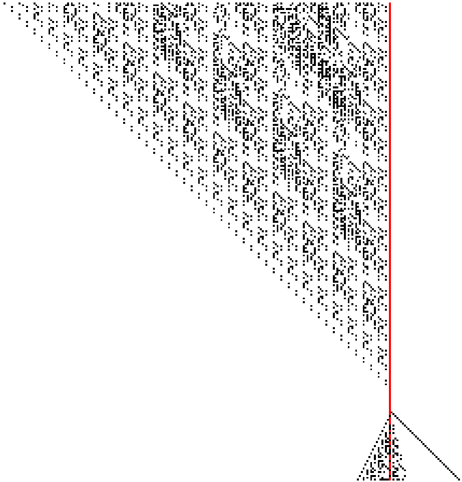

Figure : Spacetime diagram showing the beginning of the infinite history of an invertible cellular automaton containing the Rudin–Shapiro sequence.

Characterizations of automaticity that we use in our proof

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[Furstenberg, 1967] The Laurent series $F(t) = \sum_{n\geq n_0} u_n t^n$ is algebraic over $\mathbb{F}_q(t)$ if and only if it is the diagonal of a rational Laurent series in two variables over that field.

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Christol's theorem combined with Furstenberg's theorem imply that if (u_n) is p-automatic, then (u_n) can be realized as the diagonal of a quarter-lattice array of elements in $\mathbb{F}_q$ which is the formal power series expansion of $E(t,x) = \frac{P(t,x)}{Q(t,x)}$, where $P,\ Q \in \mathbb{F}_q[t,x]$.

Heuristic: Rotate this quarter array clockwise so that (u_n) shows up as a column in this diagram, and, under suitable choice of the polynomials, show that you end up with space-time diagram of a linear cellular automaton with memory.

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In particular the proof of Furstenberg's theorem implies that if (u_n) is automatic, $u_0 = 0$, $P(t, F(t))=0$ and $P_x(0,0) = \frac{\partial P(t,x)}{\partial x}|_{(0,0)} \neq 0$, then $F(t)$ is the '-2 column' of of

$$\frac{P_x(t,x)}{P(t,x)}.$$

Example: If $P(t,x) = (t^2 + t^9) + x + (t + t^2)x^2 + (t^5 + t^9)x^4$, and $(u_n)_{n\geq 0}$ is T-M, then $P(t, \sum_{n\geq 3} u_n t^{n-2}) = 0$ and $\sum_{n\geq 3} u_n t^{n-2}$ is the -2 column of

$$\frac{P_x(t,x)}{P(t,x)} = \frac{1}{x - (x - P(t,x))} = \frac{1}{x} \sum_{n\geq 0} \left(\frac{x - P(t,x)}{tx} \right)^n t^n$$

$$= \frac{1}{x} + t + \left(\frac{1}{x^2} + 1 + x \right) t^2 + \cdots$$

Sketch of proof of main result

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Now expand to get a series in t:

$$\begin{aligned}\frac{P_x(t,x)}{P(t,x)} &= \frac{P_x(t,x)}{\alpha x} \cdot \frac{1}{1 + \frac{tQ(t,x)}{\alpha x}} \\ &= \frac{P_x(t,x)}{\alpha x} \sum_{n\geq 0} \left(-\frac{Q(t,x)}{\alpha x}\right)^n t^n = \sum_{n\geq 0} R_n(x) t^n\end{aligned}$$

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$$\begin{aligned} \frac{P_x(t,x)}{P(t,x)} &= \frac{P_x(t,x)}{\alpha x} \cdot \frac{1}{1 + \frac{tQ(t,x)}{\alpha x}} \\ &= \frac{P_x(t,x)}{\alpha x} \sum_{n\geq 0} \left(-\frac{Q(t,x)}{\alpha x} \right)^n t^n = \sum_{n\geq 0} R_n(x) t^n \end{aligned}$$

As αx is a monomial, each $R_n(x)$ is a Laurent polynomial.

Sketch of proof of main result

We shall show that for some r, the shifted sequence $u_{r+1}, u_{r+2}, \ldots$ can be found as a column of a spacetime diagram of a linear cellular automaton with memory.

Suppose that $G(t) = \sum_{n\geq 1} u_{n+r} t^n$ is a root of $P(t,x) = \sum_{i=0}^{m} A_i(t) x^{p^i} + B(t) = \alpha x + tQ(t,x)$ where $\alpha \neq 0$.

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As αx is a monomial, each $R_n(x)$ is a Laurent polynomial.

It remains to show that this 2-d array is the spacetime diagram of a cellular automaton with memory.

Multiplying both sides by $P(t,x)$ gives

$$P_x(t,x) = \sum_{i=0}^{d} C_i(x)t^i \sum_{j\geq 0} R_j(x)t^j = \sum_{n\geq 0}\left(\sum_{i+j=n} C_i(x)R_j(x)\right) t^n$$
$$= \sum_{n=0}^{d}\left(\sum_{i=0}^{n} C_i(x)R_{n-i}(x)\right) t^n + \sum_{n\geq d+1}\left(\sum_{i=0}^{d} C_i(x)R_{n-i}(x)\right) t^n,$$

and since $P_x(t,x)$ is a polynomial with $\deg_t P_x(t,x) \leq d$, we have $\sum_{i=0}^{d} C_i(x)R_{n-i}(x) = 0$ for all $n \geq d+1$. Solving for $R_n(x)$ gives

$$R_n(x) = -\sum_{i=1}^{d} \frac{C_i(x)}{C_0(x)} R_{n-i}(x) = -\sum_{i=1}^{d} \frac{C_i(x)}{\alpha x} R_{n-i}(x)$$

for all $n \geq d+1$, where each $\frac{C_i(x)}{\alpha x}$ is a Laurent polynomial in x.

Technical lemma Suppose that $F(t) = \sum_{n \geq 0} u_n t^n \in \mathbb{F}_q((t))$ is algebraic over $\mathbb{F}_q(t)$. Then $G(t) \in \mathbb{F}_q((t))$ and $P(t,x) \in \mathbb{F}_q[t,x]$ of the form

$$P(t,x) = B(t) + \sum_{i=0}^{m} A_i(t) x^{p^i} = \sum_{i=0}^{d} C_i(x) t^i$$

can be computed such that

1. $F(t) = R(t) + t^r G(t)$ for some $r \geq 0$ and $R(t) \in \mathbb{F}_q[t]$,
2. $P(t, G(t)) = 0$.
3. $G(0) = 0$,
4. $C_0(x) = A_0(0)x$ is nonzero, $A_i(0) = B(0) = 0$, $1 \leq i \leq m$,
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so that (u_n) can be realized as a column of an invertible linear cellular automaton with memory. If in addition, $A_m(t)$ and $B(t)$ are monomials of degree d, then $(X_{\mathbf{u}}, \sigma)$ can be realized as a subsystem of a linear cellular automaton.

Some questions

- Each automatic sequence can be realised as a (one sided) column in an invertible cellular automaton with memory. Does every letter-to-letter projection of a bi-infinite fixed point of a length p substitution occur as a column of a bi-infinite spacetime diagram?

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- Each automatic sequence can be realised as a (one sided) column in an invertible cellular automaton with memory. Does every letter-to-letter projection of a bi-infinite fixed point of a length p substitution occur as a column of a bi-infinite spacetime diagram?
- Which k-automatic sequences (if k is not a prime power) occur as columns of cellular automaton spacetime diagrams?
- Does there exist a (non-eventually-periodic) 3-automatic sequence $(u_n)_{n\geq 0}$ on $\mathbb{F}_2$ such that (u_n) occurs as a column of a 2-state spacetime diagram? The CA rule cannot be linear over $\mathbb{F}_2$, since a sequence which is both 2-automatic and 3-automatic is eventually periodic by Cobham's theorem.