Disconnecting the G_2 moduli space

Johannes Nordström

7 July 2015

Joint work in progress with
Diarmuid Crowley and Sebastian Goette

C-N, *New invariants of G_2-structures*, arXiv:1211.0269
C-G-N, *An analytic invariant of G_2-manifolds*, arXiv:1505.02734

The G_2 moduli space

Let M be a smooth closed 7-manifold admitting metrics with holonomy G_2. The moduli space

$$\mathcal{M} := \{\text{Holonomy } G_2 \text{ metrics on } M\}/\mathrm{Diff}(M)$$

is an orbifold, locally homeomorphic to finite quotients of $H^3_{dR}(M)$.
So far little is known about the *global* properties of $\mathcal{M}$.

Main results:

Exhibit examples of closed G_2-manifolds with $\mathcal{M}$ disconnected, both

- by studying homotopies of G_2-structures, and
- where the G_2-structures are indistinguishable using homotopy theory

Outline:

- Background
- Examples
- Invariants
- Constructions
- Computation

The group G_2

$G_2 := \mathrm{Aut}\ \mathbb{O}$, $\mathbb{O} =$ octonions, normed division algebra of real dimension 8.

G_2 acts on $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$, preserving metric, orientation, cross product

$$a \times b := \mathrm{Im}(ab), \text{ and}$$
$$\varphi_0(a, b, c) := \langle a \times b, c \rangle.$$

In terms of basis $e^1, \dots, e^7 \in (\mathbb{R}^7)^*$

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \in \Lambda^3(\mathbb{R}^7)^*.$$

Peculiar algebra facts:

- G_2 is not just contained in stabiliser of φ_0 in $GL(7, \mathbb{R})$, but equality holds.
- The $GL(7, \mathbb{R})$-orbit of φ_0 is open in $\Lambda^3(\mathbb{R}^7)^*$.

G_2, spinors and $SU(3)$

$$\begin{array}{ccc} & & Spin(7) \\ & \nearrow & \downarrow \\ G_2 & \hookrightarrow & SO(7) \end{array}$$

The spin representation Δ of $Spin(7)$ is real of rank 8.

$Spin(7)$ acts transitively on $S^7 \subset \Delta$ with stabiliser G_2.

The action of $SU(3)$ on $\mathbb{C}^3 \cong \mathbb{R}^6$ preserves

$$\omega_0 := \tfrac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3) \in \Lambda^2(\mathbb{R}^6)^*$$

$$\Omega_0 := dz^1 \wedge dz^2 \wedge dz^3 \in \Lambda^3(\mathbb{R}^6)^* \otimes \mathbb{C}$$

On $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$,

$$\varphi_0 \ = \ e^1 \wedge (e^{23} + e^{45} + e^{67}) + e^{246} - e^{257} - e^{347} - e^{356} \ \cong \ e^1 \wedge \omega_0 + \mathrm{Re}\,\Omega_0$$

and the stabiliser in G_2 of a non-zero vector is $SU(3)$.

G_2-structures and holonomy

A 3-form $\varphi \in \Omega^3(M^7)$ such that $(T_xM, \varphi) \cong (\mathbb{R}^7, \varphi_0)$ for all $x \in M$ defines a G_2-structure. (*Open* condition on φ)
Because $G_2 \subset SO(7)$, this induces a metric and orientation.

The holonomy group of a Riemannian manifold M

$$\{P_\gamma : \gamma \text{ closed loop based at } x \in M\} \subseteq O(T_xM)$$

where P_γ denotes parallel transport along γ.
Parallel tensor fields on $M \leftrightarrow$ invariants of $Hol(M)$.

$Hol(M) \subseteq G_2 \Leftrightarrow$ metric induced by some G_2-structure φ such that $\nabla\varphi = 0$.
Then call φ *torsion-free*. This is equivalent to the first-order non-linear PDE

$$d\varphi = d^*\varphi = 0$$

Proposition (Joyce)

If M^7 is closed and $Hol(M) \subseteq G_2$ then

$$Hol(M) = G_2 \Leftrightarrow \pi_1(M) \textit{ finite}$$

Two perspectives on G_2-structures

G_2	$=$	stabiliser in $GL(7,\mathbb{R})$ of $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$	$=$	stabiliser in $Spin(7)$ of a unit spinor s_0
G_2-structure on M^7	$\leftrightarrow$	positive $\varphi \in \Omega^3(M)$	$\leftrightarrow$	metric g $+$ spin structure $+$ unit spinor field s
Holonomy $\subseteq G_2$	$\Leftrightarrow$	$d\varphi = d^*\varphi = 0$	$\Leftrightarrow$	$\nabla s = 0$
Useful for		differential geometry		homotopy theory

Homotopies of G_2-structures

Let M be a closed 7-dimensional spin manifold.
All metrics on M are homotopic.
Two G_2-structures homotopic if connected by path of non-vanishing spinors.

Homotopy classes of G_2-structures on M	$\leftrightarrow$	Homotopy classes of non-vanishing sections of SM

The spinor bundle SM is a real rank 8 vector bundle. Easy consequences:

- There exist G_2-structures on M.
- For G_2-structures φ and φ' on M, the signed count of zeros of interpolating section of rank 8 bundle on $M \times [0,1]$ can take any integer value, and vanishes if and only if φ is homotopic to φ'.

$$\therefore \{G_2\text{-structures on } M\}/\text{homotopy} \overset{\text{affine}}{\cong} \mathbb{Z}$$

Diff(M) can act by non-trivial translations. Each component of $\mathcal{M}$ maps to a fixed class of G_2-structures modulo homotopies *and* diffeomorphisms.

2-connected 7-manifolds

Let M be a closed smooth 7-manifold with $\pi_1(M) = \pi_2(M) = 0$ and $H^4(M)$ torsion-free. Remaining algebraic topology captured by $b_3(M)$.
Let $d(M) :=$ greatest integer dividing the Pontrjagin class $p_1(M) \in H^4(M)$ ($d(M) := 0$ if $p_1(M) = 0$).

Theorem (Wall-Wilkens)

Such M are classified up to homeomorphism by $(b_3(M), d(M)) \in \mathbb{N} \times 4\mathbb{N}$. The number of inequivalent smooth structures on the topological manifold underlying M is

$$\mathrm{GCD}\left(28,\ \mathrm{Numerator}\left(\frac{d(M)}{8}\right)\right).$$

Theorem (C-N)

The number of G_2-structures on M modulo homotopy and diffeomorphism is

$$24\,\mathrm{Numerator}\left(\frac{d(M)}{224}\right).$$

Examples

Example A (C-G-N) $b_3 = \ 97, \ d = 4$

There are G_2 metrics on M whose associated G_2-structures are not equivalent under homotopies and diffeomorphisms. Thus $\mathcal{M}$ is disconnected.

Example B (C-G-N) $b_3 = 109, \ d = 4$

There are G_2 metrics on M that lie in different components of $\mathcal{M}$, but whose associated G_2-structures are homotopic.

Side remark:
Example B shows that there is no h-principle for torsion-free G_2-structures (would have been surprising for an essentially elliptic equation).
However, the h-principle holds for coclosed G_2-structures (C-N).

Ingredients

Invariants

A The G_2-structures are distinguished by a homotopy invariant $\nu(\varphi) \in \mathbb{Z}/48\mathbb{Z}$.

B An analytic refinement $\widehat{\nu}(\varphi) \in \mathbb{Z}$ of $\nu(\varphi)$ is invariant under deformations through torsion-free G_2-structures, and can distinguish components of $\mathcal{M}$ even when the G_2-structures are homotopic.

Twisted connected sums

The "twisted connected sum construction" of Kovalev and Corti-Haskins-N-Pacini produces large numbers of 2-connected G_2-manifolds for which these invariants can be evaluated. However, $\widehat{\nu}$ is always -24.

A more complicated version produces some 2-connected examples where $\widehat{\nu}$ takes different values.

Homotopy invariant of G_2-structures

Let X closed spin 8-manifold. Euler class of positive spinor bundle satisfies

$$e_+(X) = 24\widehat{A}(X) + \frac{\chi(X) - 3\sigma(X)}{2}, \qquad (*)$$

where χ is the Euler characteristic and σ the signature.

Let W be a compact spin 8-manifold with boundary M, s a transverse positive spinor field on W, and φ the G_2-structure on M induced by $s_{|M}$. Let $n(W, \varphi)$ be the signed count of zeros of s. $(*)$ implies

$$\nu(\varphi) := \chi(W) - 3\sigma(W) - 2n(W, \varphi) \mod 48$$

is independent of choice of coboundary W.
On a fixed M, ν takes 24 values allowed by $\nu(\varphi) = \sum_{i=0}^{3} b_i(M) \mod 2$.

Corollary (C-N)

Let M closed 2-connected with $H^4(M)$ torsion-free. If $d(M) \mid 224$ then ν classifies G_2-structures on M modulo homotopies and diffeomorphisms.

Analytic invariant of G_2-structures

Given metric, define

$D =$ Dirac operator
$B : \Omega^{ev} \to \Omega^{ev} =$ odd signature operator, $(-1)^k(*d - d*)\ \Omega^{2k}$
$h(D) = \dim\ker(D) \in \mathbb{Z}$
$\eta(D) := \eta(D, 0) \in \mathbb{R}$ measures "spectral asymmetry" of D, defined by analytic continuation from

$$\eta(D, s) := \sum_{\lambda \in \mathrm{Spec} D \setminus \{0\}} (\mathrm{sign}\lambda)|\lambda|^{-s} \quad \text{for } \operatorname{Re} s >> 0$$

For a G_2-structure φ on closed M^7, define $MQ(\varphi) \in \mathbb{R}$ in terms of "Mathai-Quillen current".

Definition

$$\widehat{\nu}_0(\varphi) := -24\eta(D) + 3\eta(B) + 2MQ(\varphi) \in \mathbb{R}$$

$$\widehat{\nu}(\varphi) := \widehat{\nu}_0(\varphi) - 24h(D) \in \mathbb{R}$$

Analytic invariant as refinement

$$\widehat{\nu}_0(\varphi) := -24\eta(D) + 3\eta(B) + 2MQ(\varphi) \in \mathbb{R}$$

Reversing orientation changes the sign of $\widehat{\nu}_0$.

All terms are continuous in φ, except that the first jumps by 24 when an eigenvalue of D changes between zero and non-zero.

$$\widehat{\nu}(\varphi) := \widehat{\nu}_0(\varphi) - 24h(D) \in \mathbb{R}$$

$\widehat{\nu}$ is continuous in φ except for jumps by 48.

Theorem (C-G-N)

Let φ G_2-structure on closed M^7. Then

$$\nu(\varphi) = \widehat{\nu}(\varphi) \mod 48.$$

(In particular $\widehat{\nu}, \widehat{\nu}_0 \in \mathbb{Z}$.)

Analytic invariant as refinement

$$\widehat{\nu}(\varphi) := -24(\eta + h)(D) + 3\eta(B) + 2MQ(\varphi) \in \mathbb{R}$$
$$\nu(\varphi) := \chi(W) - 3\sigma(W) - 2n(W, \varphi) \in \mathbb{Z}/48\mathbb{Z}.$$

Proof.

For $\partial W = M$ with metric that is product on collar of M

$$\begin{array}{rcccl} \sigma(W) & = & \int_W L(\nabla) & - & \eta(B) \\ \operatorname{ind} D_W^+ & = & \int_W \widehat{A}(\nabla) & - & \frac{1}{2}(\eta + h)(D) \\ n(W, \varphi) & = & \int_W e_+(\nabla) & - & MQ(\varphi) \\ & & \text{Chern-Weil term} & & \text{boundary correction} \end{array}$$

Chern-Weil terms add up to $\chi(W)$ (essentially by characteristic class formula $(*)$ used to show that ν is well-defined), so

$$\widehat{\nu}(\varphi) = \chi(W) - 3\sigma(W) - 2n(W, \varphi) + 48 \operatorname{ind} D_W^+ \in \mathbb{Z}\,.$$

□

Analytic invariant of torsion-free G_2-structures

$$\widehat{\nu}_0(\varphi) := -24\eta(D) + 3\eta(B) + 2MQ(\varphi) \in \mathbb{Z}$$
$$\widehat{\nu}(\varphi) := \widehat{\nu}_0(\varphi) - 24h(D) \in \mathbb{Z}$$

For torsion-free φ

- $MQ(\varphi) = 0$
- $h(D) = 1 + b_1(M)$ (so 1 when $Hol = G_2$)
- $\eta(D)$ does not jump

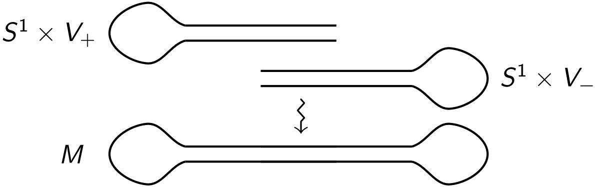
Therefore $\widehat{\nu}_0$ and $\widehat{\nu}$ are constant on connected components of $\mathcal{M}$, and can distinguish components even when the associated G_2-structures are homotopic.

Even if we are only interested in ν (like in Example A), it may be easier to evaluate the intrinsic formula for $\widehat{\nu}$ than to find a spin coboundary to compute ν.

Twisted connected sums

Donaldson, Kovalev, Corti-Haskins-N-Pacini

- Construct simply-connected, complete, Ricci-flat Kähler 3-folds V, with "asymptotically cylindrical end" $\mathbb{R} \times S^1 \times K3$.
- $\mathrm{Hol}(S^1 \times V) = \mathrm{SU}(3) \subset G_2$, so $S^1 \times V$ has torsion-free G_2-structure
- Find pairs of such $V_\pm$, with a diffeomorphism F of the cylindrical ends of $S^1 \times V_+$ and $S^1 \times V_-$ ensuring
  - $M = S^1 \times V_+ \cup_F S^1 \times V_-$ is simply-connected (F is "twisted")
  - Gluing G_2-structures on the halves with "neck length" $T >> 0$ defines φ_T on M with $\nabla\varphi_T$ exponentially small in T.



- Perturb to φ_T so that $d\varphi_T = d^*\varphi_T = 0$. Then $\mathrm{Hol}(M) = G_2$.

Matching

The ACyl end of $S^1 \times V_\pm$ is $\mathbb{R} \times S^1 \times S^1 \times K3_+ \cong \mathbb{R} \times T^2_\pm \times K3_\pm$.
Glue the cylindrical ends using a product isometry

$$F := (-1) \times m \times r : \mathbb{R} \times T^2_- \times K3_+ \to \mathbb{R} \times T^2_- \times K3_-,$$

where $m : T^2_+ \to T^2_-$ is the reflection $S^1 \times S^1 \to S^1 \times S^1$, $(u, v) \mapsto (v, u)$.
m swaps "internal" and "external" circles $\Rightarrow \pi_1 M = 0$ by van Kampen.

Matching problem: Find pairs V_+ and V_- such that there is an isometry $r : K3_+ \to K3_-$ making F an isomorphism of the ACyl G_2-structures.

Kovalev:
Use Fano 3-folds to produce examples of pairs V_+, V_- with solution to the matching problem.

Corti-Haskins-N-Pacini:
Millions of examples from weak Fano 3-folds.
Topological type determined in many cases.
Many gluings give same smooth manifold.

Invariants of twisted connected sums

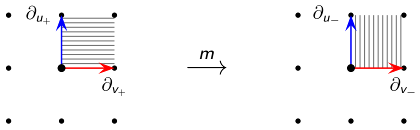
Theorem (C-N)

Any twisted connected sum has $\nu = 24 \in \mathbb{Z}/48\mathbb{Z}$*.*

Theorem (C-G-N)

Any twisted connected sum has $\widehat{\nu} = -24 \in \mathbb{Z}$*.*

Related geometric feature:
$m : T^2_+ \to T^2_-$ aligns "external" circle tangents ∂_v at right angle.

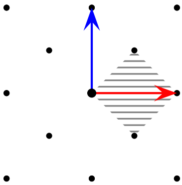


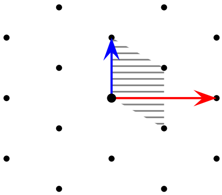
Inevitable, because m is an isometry of rectangular tori, and is not allowed to align the external circles: otherwise M would have an S^1 factor.

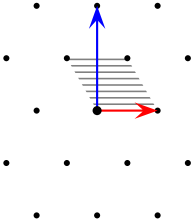
Tori with symmetries

Let $a\colon S^1 \to S^1$ be the antipodal map $z \mapsto -z$.
Let $T^2 := S^1 \times S^1 \;/\; a \times a$ where the S^1 factors have circumference 1 and x.
For how many different x does T^2 have rotation symmetries other than ± 1?

$$x = 1, \sqrt{3}, \text{ or } \frac{1}{\sqrt{3}}$$







Extra-twisted connected sums

Suppose V is an ACyl Calabi-Yau with an involution τ, that acts on the asymptotic cross-section $S^1 \times K3$ by $a \times \mathrm{Id}_{K3}$
Then $S^1 \times V \,/\, a \times \tau$ is an ACyl G_2-manifold with cross-section

$$(S^1 \times S^1 \,/\, a \times a) \times K3 = T^2 \times K3.$$

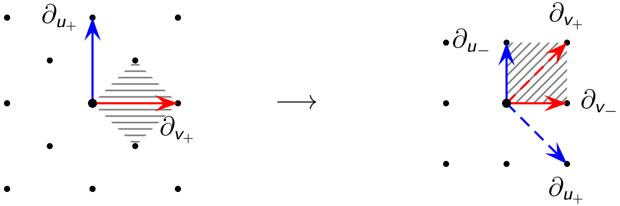
Let $M_\pm$ be a pair of ACyl G_2-manifolds of this form, or of the form $S^1 \times V$.
Let $m : T^2_+ \to T^2_-$ be a reflection. Depending on the circumferences of the circles, the external circle directions can be aligned at angle $\theta = \frac{\pi}{3}$, $\frac{\pi}{4}$ or $\frac{\pi}{6}$.

θ-matching problem: Find pairs V_+ and V_- with involution, and with an isometry $r : K3_+ \to K3_-$ such that $(-1) \times m \times r$ an isomorphism of the ACyl G_2-structures of M_+ and M_-.

Some examples can be found from branched double covers of Fano 3-folds.

Extra-twisted connected sums

Can achieve $\theta = \frac{\pi}{4}$ with an involution on one side.

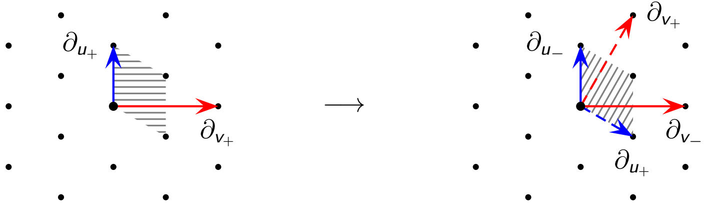


Example A:

Use classification of 2-connected 7-manifolds to identify a certain $\frac{\pi}{4}$-TCS that has $\nu = 36 \in \mathbb{Z}/48\mathbb{Z}$ with an ordinary TCS.
The latter has $\nu = 24 \in \mathbb{Z}/48\mathbb{Z}$, so the G_2-structures are not homotopic.

Extra-twisted connected sums

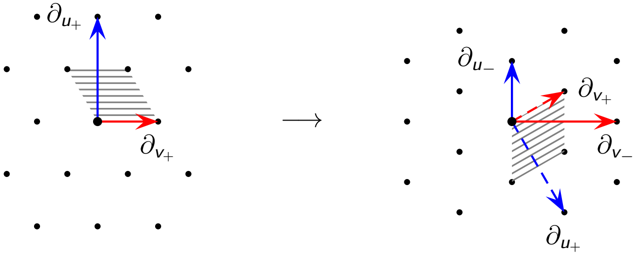
With involutions on both sides, one can achieve $\theta = \frac{\pi}{3}$.



These examples have 3-torsion in $H^4(M)$, making it harder to apply classification results to find different examples realising the same smooth manifold.

Extra-twisted connected sums

With involutions on both sides, one can achieve $\theta = \dfrac{\pi}{6}$.



Example B:

Use classification of 2-connected 7-manifolds to identify a certain $\frac{\pi}{6}$-TCS with $\widehat{\nu} = -72$ with an ordinary TCS.

Both G_2-structures have $\nu = 24 \in \mathbb{Z}/48\mathbb{Z}$, and on this manifold ν classifies G_2-structures up to homotopy.

Computing the eta invariants

$M_\pm := S^1 \times V_\pm$ or $S^1 \times V_\pm / a \times \tau$, with asymptotic limit $\mathbb{R} \times T^2_\pm \times K3$.
$m : T^2_+ \to T^2_-$ reflection, aligning external circle factors at angle $\theta \in (0, \frac{\pi}{2}]$.
Construct family of torsion-free G_2-structures φ_T with "neck length" T on M the result of gluing by $(-1) \times m \times r$.

Theorem
Let $\rho := \pi - 2\theta$. Then $\eta(D) \to \frac{\rho}{\pi}$ as $T \to \infty$.

Let $N_\pm := \operatorname{Im}(H^2(V_\pm) \to H^2(K3))$, and $R_{N_\pm} : H^2(K3;\mathbb{R}) \to H^2(K3;\mathbb{R})$ the reflection in $N_\pm$ (using L^2-metric or intersection form gives same result!)

Theorem
Define a unitary map $U : H^2(K3;\mathbb{C}) \to H^2(K3;\mathbb{C})$ by $e^{\pm i\rho} R_{N_+} R_{N_-}$ on $H^{2,\pm}(K3;\mathbb{C})$. Then

$$\eta(B) \to \frac{1}{\pi} \sum_{\substack{\lambda \in \operatorname{Spec} U \\ \lambda \neq -1}} \arg \lambda$$

as $T \to \infty$, where the branch of arg *takes values in $(-\pi, \pi)$.*

Evaluating $\widehat{\nu}$

$U := e^{\pm i\rho} R_{N_+} R_{N_-}$ on $H^{2,\pm}(K3;\mathbb{C})$. The theorems imply

$$\widehat{\nu}_0 = -24\eta(D) + 3\eta(B) = -24\frac{\rho}{\pi} + \frac{3}{\pi} \sum_{\substack{\lambda \in \mathrm{Spec}\, U \\ \lambda \neq -1}} \arg \lambda.$$

If $\theta = \frac{\pi}{2}$ then $\rho = \pi - 2\theta = 0$, and U is the real orthogonal map $R_{N_+} R_{N_-}$. Hence eigenvalues are ± 1 or occur in conjugate pairs, so $\sum \arg \lambda = 0$, and

$$\widehat{\nu}_0 = 0.$$

In general

$$\sum_{\substack{\lambda \in \mathrm{Spec}\, U \\ \lambda \neq -1}} \arg \lambda \;=\; \sum \pm i\rho \;+ \sum_{\substack{\lambda \in \mathrm{Spec}\, R_{N_+} R_{N_-} \\ \lambda \neq -1}} \arg \lambda \;+\; b \;=\; -16\rho + \pi\, b,$$

where $b \in \mathbb{Z}$ counts "half branch jumps" between λ and $e^{\pm i\rho}\lambda$. Then

$$\widehat{\nu}_0 = -72\frac{\rho}{\pi} + 3b.$$

Sketch proof of theorem for $\eta(B)$

Kirk-Lesch gluing formula:

$$\eta(B) \to \eta(B_+) + \eta(B_-) + \text{Maslov index}$$

as $T \to \infty$, for $B_\pm$ the odd signature operators on manifolds with boundary.

Because $M_\pm$ have an S^1-factor they have an orientation-reversing isometry. Therefore $B_\pm$ has spectral symmetry, so $\eta(B_\pm) = 0$!

Consider $H^3(T^2 \times K3)$ as a complex vector space, with complex structure $*$. The Maslov index is computed in terms of the spectrum of $-R_+R_-$, where $R_\pm$ is reflection of $H^3(T^2 \times K3)$ in the image of $H^3(M_+)$.
$H^3(T^2 \times K3) \cong H^1(T^2) \otimes H^2(K3) \cong \mathbb{C} \otimes H^{2,+}(K3) \oplus \overline{\mathbb{C}} \otimes H^{2,-}(K3)$.
$R_\pm \cong R_{\partial_{u_\pm}} \otimes R_{N_\pm}$.
$-R_{\partial_{u_+}} R_{\partial_{u_-}}$ is rotation by $\rho = \pi - 2\theta \rightsquigarrow$
$-R_+R_- \cong e^{\pm i\rho} R_{N_+} R_{N_-}$ on $H^{2,\pm}(K3; \mathbb{C})$.