Strong shift equivalence of matrices over a ring

joint work in progress with Scott Schmieding

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Introduction

For problems of $\mathbb{Z}^d$ SFTs and their relatives:

$d \geq 2$:
computability conditions are fundamental.

$d = 1$:
Key features:
(1) Algebra around matrices
(SSE, SE, related invariants)
(2) Positivity constraints

This talk reports progress on (1).

All rings and semirings are assumed to contain $\{0, 1\}$.

Strong shift equivalence

Let $\mathcal{S}$ be a semiring.
And on the first day [1973], Williams defined strong shift equivalence.

Matrices A, B over $\mathcal{S}$ are elementary strong shift equivalent over $\mathcal{S}$ (ESSE-$\mathcal{S}$) if they are square and there exist matrices U, V over $\mathcal{S}$ such that

$$A = UV \quad \text{and} \quad B = VU \ .$$

A, B are strong shift equivalent over $\mathcal{S}$ (SSE-$\mathcal{S}$) if there exists a chain

$$A = A_0, A_1, \ldots, A_\ell = B$$

with A_{i-1} and A_i ESSE-$\mathcal{S}$ for $0 < i \le \ell$.

Why did Williams define SSE?

- Up to topological conjugacy, every shift of finite type (SFT) is an "edge SFT" σ_A, defined by a square matrix A over $\mathbb{Z}_+$.

- σ_A and σ_B are isomorphic (topologically conjugate) iff A, B are SSE-$\mathbb{Z}_+$.

But SSE over $\mathbb{Z}_+$ is very hard to understand completely (not known to be decidable, even restricted to small cases).

So on the second day, Williams defined ...

Shift equivalence

DEFN Square matrices A, B are shift equivalent over $\mathcal{S}$ (SE-$\mathcal{S}$) if $\exists$ matrices U, V over $\mathcal{S}$ and $\ell \in \mathbb{N}$ such that

$$A^\ell = UV \qquad B^\ell = VU$$
$$AU = UB \qquad BV = VA$$

Always: SSE-$\mathcal{S}$ implies SE-$\mathcal{S}$. Also

- SE-$\mathbb{Z}_+$ is decidable (Kim-Roush).

- SE-$\mathbb{Z}_+$ turns out to be reasonably tractable, and closely related to significant applications in symbolic dynamics

- SE over $\mathbb{Z}$ (or other rings) has useful and conceptually satisfying algebraic reformulations.

Classifying shifts of finite type.
Williams gave us:

- Theorem (Annals of Math 1973)
  SE-$\mathbb{Z}_+$ $\Longrightarrow$ SSE-$\mathbb{Z}_+$.

- Conjecture (Annals of Math 1974)
  SE-$\mathbb{Z}_+$ $\Longrightarrow$ SSE-$\mathbb{Z}_+$.

Eventually counterexamples were constructed (Kim Roush 1992,1999), based on a lovely algebraic topological structure created by Wagoner ("strong shift equivalence space").
No progress since on understanding refinement of SSE-$\mathbb{Z}_+$ by SE-$\mathbb{Z}_+$.
However ...
From here $\mathcal{S}$ is a ring.

There are good reasons to study SSE over other rings and semirings.

- To approach the $\mathbb{Z}$ problem.

- There are other symbolic dynamical systems presented by matrices over $\mathcal{S}_+$ and classified up to conjugacy by SSE over $\mathcal{S}_+$. E.g.:

  $\mathcal{S} = \mathbb{Z}G$, G finite:
  SSE-$\mathbb{Z}_+G$ classifies free G-SFTs.

  $\mathcal{S} = \mathbb{Z}G$, $G = \mathbb{Z}^n$: SSE-$\mathbb{Z}_+G$ classifies irred. SFTs with Markov measure.

  $\mathcal{S}$ = integral semigroup ring of a certain noncommutative semigroup:
  SSE over $\mathcal{S}_+$ classifies sofic shifts.

- For understanding constraints of order on algebraic properties of matrices.

- Understanding SSE-$\mathcal{S}$ for its own sake.

- Understand better proofs that can't work and theorems that can't be proved.

Before confronting the hard problem of understanding how SSE-$\mathcal{S}_+$ refines SE-$\mathcal{S}_+$, we would like to understand how SSE-$\mathcal{S}$ refines SE-$\mathcal{S}$.

It was known that SE-$\mathcal{S}$ implies SSE-$\mathcal{S}$ if
$\mathcal{S} = \mathbb{Z}$ (Williams, 70s)
$\mathcal{S} =$PID (Effros, 80s)
$\mathcal{S} =$Dedekind domain (B-Handelman,90s).

That was it.

Definitions

GL($\mathcal{S}$) = group of $\mathbb{N} \times \mathbb{N}$ matrices $\begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$ with U finite invertible.

EL($\mathcal{S}$) = subgroup generated by basic elementary matrices E
($E = I$ except perhaps in one offdiagonal entry)

EL($\mathcal{S}$) = commutator subgroup

$K_1(\mathcal{S}) = \text{GL}(\mathcal{S})/\text{EL}(\mathcal{S})$

The central connection for clarifying SSE-$\mathcal{S}$ is ...

THEOREM (B-Schmieding)
Suppose A, B are matrices over $\mathcal{S}$. TFAE.

(1) A and B are SSE over $\mathcal{S}$.

(2) There are E, F in El($\mathcal{S}[t]$) such that
$E(I - tA)F = (I - tB)$.

The finite matrices $I - tA, I - tB$ are embedded as the upper left corners of matrices with all other entries zero (and identified with these infinite matrices).

This grows out of work by Shannon, BGMY, Wagoner, Kim-Roush-Wagoner, B-Sullivan.

The theorem above leads to ...

THEOREM (B-Schmieding)
Let A be a square matrix over $\mathcal{S}$.

(I) If B is SE over $\mathcal{S}$ to A, then there is a nilpotent matrix N such that

$$\begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix}$$

is SSE over $\mathcal{S}$ to B.

(II) The map

$$\begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix} \to I - tN$$

induces a bijection from the set of SSE classes of matrices SE over $\mathcal{S}$ to A to the abelian group

$$NK_1(\mathcal{S})/H_A \ .$$

The group $NK_1(\mathcal{S})$ is an important group in the algebraic K-theory of the ring $\mathcal{S}$. It is the kernel of the map

$$K_1(\mathcal{S}[t]) \to K_1(\mathcal{S})$$

induced by $t \mapsto 0$.

The group H_A is the set of elements in $K_1(\mathcal{S})$ containing a matrix U
such that
there is E in $\mathrm{El}(\mathcal{S})$ such that
$U(I - tA)E = I - tA$.

What about this group

$$NK_1(\mathcal{S})/H_A$$

which captures the refinement of SE-$\mathcal{S}$ by SSE-$\mathcal{S}$?

$NK_1(\mathcal{S})$ if nontrivial is not finitely generated (Farrell 1977).

$H_A = 0$ if A is nilpotent or $\mathcal{S}$ is commutative.

Any consequences of Theorem?

Known fact: for $\mathcal{S} = ZG$ with $G = Z/nZ$: $NK_1(\mathcal{S}) = 0$ iff n is squarefree.

For the not-squarefree case: we expect this will let us refute a working conjecture of Bill Parry on the classification of skew products of mixing SFTs by finite groups.

For a huge class of rings, we now know SE-$\mathcal{S}$ implies SSE-$\mathcal{S}$. This includes $\mathbb{Z}G$ with $G = \mathbb{Z}^n$.

THM. Suppose A and B are matrices over a dense subring $\mathcal{S}$ of the reals, with A primitive and B SE over $\mathcal{S}$ to A, with trace$(A) > 0$. Then B is SSE over $\mathcal{S}$ to a primitive matrix.

(The "Generalized Spectral Conjecture" of B-Handelman is reduced to realization by any element of a shift equivalence class.)

In "Path Methods for strong shift equivalence of positive matrices" (B-Kim-Roush 2013), the constructions of certain SSEs of positive matrices A, B over $\mathcal{S}$ a dense subring of R depended on an assumption A, B SSE over $\mathcal{S}$ (not just SE). We now know this is not an artifact of a deficient proof. E.g., $\mathcal{S} = \mathbb{Q}[\pi^2, \pi^3, e, e^{-1}]$ has $NK_1(\mathcal{S})$ nontrivial.

In (B-Kim-Roush 2013), a 3-step program was proposed for understanding SSE-$\mathcal{S}_+$ of positive trace matrices over $\mathcal{S}$ a dense subring of $\mathbb{R}$. One step was to understand the refinement of SE by SSE over $\mathcal{S}$.

In this work, we found a characterization of equivalence in the Bass group $\mathrm{Nil}_0(\mathcal{S})$ which (so far?) we have not found in the literature.

The connections involved in these results may lead to ideas useful for understanding the $\mathbb{Z}_+$ case of SSE. This suggestion is perhaps not so wild as it might appear.

As Sinai replied, when asked after a talk whether he thought his probabilistic approach to the Mobius subshift could lead to a proof of the Riemann Hypothesis:

The situation is not hopeless.