WCATSS Lecture 3.3: Khovanov homology

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1 Khovanov homology generalizes the Jones polynomial

Khovanov homology is obtained by the following process:

knot / link L with projection in $\mathbf{R}^2$ $\longrightarrow$ cube of circles (“smoothings”) in $\mathbf{R}^2$ $\longrightarrow$ chain complex of graded vector spaces $\longrightarrow$ Khovanov homology

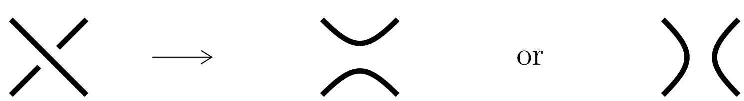
Roughly, Khovanov homology generalizes the Jones polynomial in that its Euler characteristic is the Jones polynomial. In more detail, a chain complex of *graded* vector spaces has a *graded* Euler characteristic, which is a polynomial rather than a number. This is because a graded vector space $V = \bigoplus_i V_i$, has a graded-dimension, defined as

$$\mathrm{qdim}(V) = \sum_i q^i \dim(V_i).$$

In the process shown above, we associate a certain graded vector space to each smoothing. The homology of the complex is the Khovanov homology, and its graded Euler characteristic is the unnormalized Jones polynomial $\widehat{J}(L)$.

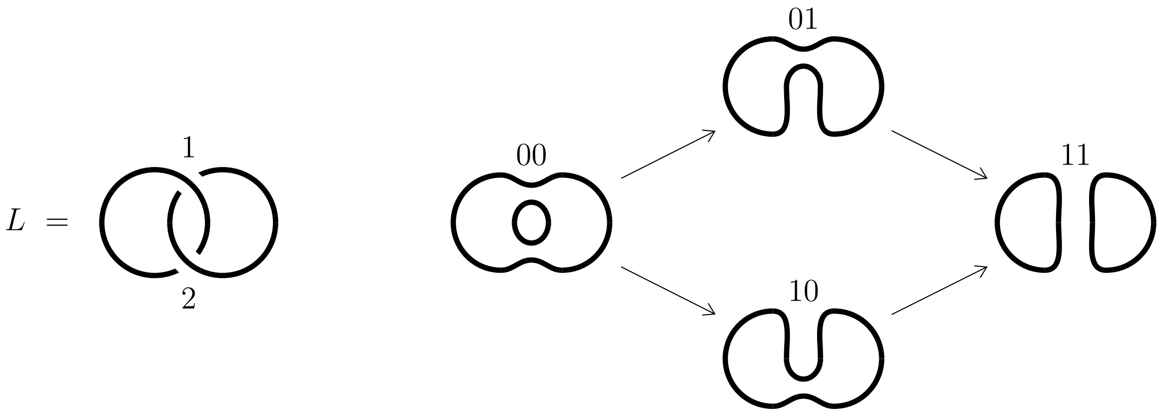
One can slightly generalize the above process by replacing chain complexes of graded vector spaces by chains of objects in a cobordism category, and then obtaining the rightmost arrow by the TQFT underlying Khovanov homology. The most immediate concrete payoff is an extension from knots and links to tangles, but we also get the conceptual benefit of being able to view Kh as a functor between two cobordism categories with some extra structure (which we define later).

First recall the types of smoothings of a crossing:



0-smoothing 1-smoothing

As in the Jones polynomial lecture, we calculate the cube of circles for the Hopf link, with an ordering on the crossings. Start with all 0-smoothings on the left-most diagram, and with each column to the right, increase the number of 1-smoothings by 1.



Each vertex of the cube is called a *smoothing* and denoted s_α for the α as written above each smoothing. If we have an n-crossing link, then the cube of smoothings is indexed by $\alpha \in \{0,1\}^n$. For a fixed smoothing, set

$$k := \text{number of circles in } S^1,$$
$$r = |\alpha| := \text{number of 1-smoothings in } \alpha.$$

where r is called the *height* of α. The Jones polynomial $\widehat{J}(L)$ was defined by associating a monomial m_α to each smoothing s_α:

$$m_\alpha := (-1)^r q^r (q^{-1} + q)^k,$$
$$\widehat{J}(L) := \left(\sum_\alpha m_\alpha \right) (-1)^{n_-} q^{n_+ - 2n_-},$$

Here n_+ the number of positive crossings in the oriented link diagram, and n_- the number of negative link crossings, with

positive crossing, negative crossing.

For the Khovanov homolgy, each smoothing s_α has an associated vector space $V_\alpha = V^{\otimes k}$, or more accurately, $V^{\otimes k}\{r\}$, where $\{r\}$ is the *grading shift*. The space V is a graded vector space with one basis element v_+ of grading $+1$ and one basis element v_- of grading -1. One defines a chain complex by setting the r-th term to be $\bigoplus_{\alpha : |\alpha| = r} V_\alpha$. See Diagram (3) in Section 3.2 of Bar-Natan's paper `arxiv:0201043` for an example of calculating these (so far ungraded) chain groups in the chain complex $[\![L]\!]$ and graded chain complex $C(L)$.

The differential is a direct sum of maps, one for each edge joining vertices of height r to vertices of height $r-1$. Each of these maps is obtained by drawing a cobordism for each such edge and applying a TQFT. Notice that the smoothings joined by edges differ by either "merging two circles" or "splitting one circle into two." In these two cases respectively, draw the "pants" (upside-down pants) and "co-pants" (right-side-up pants) cobordisms (going from top to bottom). The map between V_α's is then determined by applying the 2-TQFT (or Frobenius algebra) given by $H^*(\mathbf{CP}^1)$, where

$$v_+ \longleftrightarrow 1 \in H^*(\mathbf{CP}^1) \quad , \quad v_- \longleftrightarrow x \in H^*(\mathbf{CP}^1).$$

2 Isotopy invariance

Theorem 1. *The homology of this $C^*(L)$ depends only on the isotopy class of L. That is, it is independent of the projection to $\mathbf{R}^2$.*

The proof of this theorem is not given. Instead, we sketch a proof of the generalization, given by Theorem 2. We end the discussion by noting that the Jones polynomial can be recovered from the Poincaré polynomial of $H^*(C(L))$,

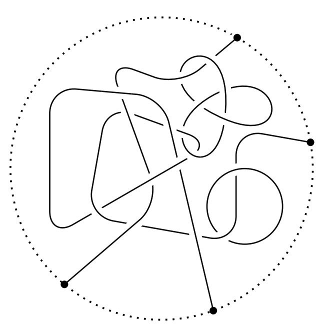
$$\sum_{r=-\infty}^{\infty} t^r \mathrm{qdim}(H^r(C(L))).$$

In fact, plugging in $t = -1$, we get

$$\sum_{r=-\infty}^{\infty} (-1)^r \mathrm{qdim}(H^r(C(L))) = \widehat{J}(L).$$

3 Extending to tangles

We will soon generalize from links to *tangles*, which are embeddings of a (not necessarily connected) compact 1-dimensional manifold, now *possibly with boundary*, into D^3. When taking a projection to $\mathbf{R}^2$, we require the boundary of the 1-manifolds to lie on $S^1 = \partial D^2$ in some specific arrangement of points $B \subset S^1$. We consider tangles up to isotopies that are the identity on ∂D^3. Here is an example of a tangle diagram:

$T =$ 

Smoothings of tangle diagrams are defined similarly to smoothings of link diagrams.

4 Categories and bordisms

Definition 1. *A category C is called pre-additive if* $\mathrm{Mor}(A, B)$ *is an abelian group for all* $A, B \in \mathrm{Ob}(C)$ *and if composition of morphisms is bilinear.*

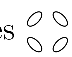
We can make any category pre-additive by taking formal sums of morphisms. Next, let C be pre-additive and define the following categories.

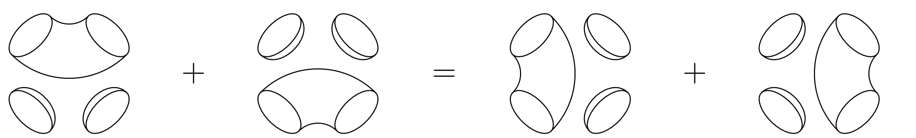
category	**objects**	**morphisms**
$\mathcal{C}ob^3(B)$	tangle smoothings with boundary B	bordisms between such smoothings
$\mathcal{M}at(C)$	$\bigoplus_{i=1}^n \mathcal{O}_i$ for all $\mathcal{O}_i \in \mathrm{Ob}(C)$	$F : \bigoplus^n \mathcal{O}_i \to \bigoplus^m \mathcal{O}'_i$ with $F_{jk} : \mathcal{O}_k \to \mathcal{O}'_j$
$\mathcal{K}om(C)$	$\cdots \xrightarrow{d^{r-1}} \Omega^r \xrightarrow{d^r} \Omega^{r+1} \xrightarrow{d^{r+1}} \cdots$ with $d^{r+1}d^r = 0$ for all r	$F = (F_r)$ that commutes with the differentials d^r

Define $\mathcal{K}ob(B) := \mathcal{K}om(\mathcal{M}at(\mathcal{C}ob^3(B)))$, and note that the cube $[\![T]\!]$ is an object in $\mathcal{K}ob(B)$. Sometimes we suppress the dependence on B in the notation. Next, to get an invariant of tangles, we need to mod out by chain homotopies of complexes on objects of $\mathcal{K}om$ and "local relations" on morphisms of $\mathcal{C}ob^3$. The local relations are defined below.

S: If a bordism has an S^2 component, set the bordism to 0.

T: If a bordism has a torus component, remove the torus and multiply the bordism by 2.

$4Tu$: If the intersection of a bordism and a 2-sphere is four circles , then



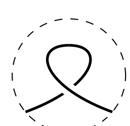
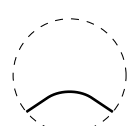
where the bordism outside the diagram of each term is the same.

These relations come from the Frobenius algebra given by $H^*(S^2)$ (see the last section of this note). In fact, S can be deduced by decomposing the sphere as a cap and a cup, and T can be deduced from decomposing the torus as a cap, co-pants, pants, and cup.

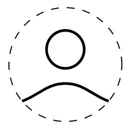
Let $\mathcal{K}om_{/h}(C)$ be the category $\mathcal{K}om(C)$ modulo chain homotopies of objects, $\mathcal{C}ob^3_{/\ell}(B)$ the category $\mathcal{C}ob^3(B)$ modulo the S, T, and $4Tu$ relations of morphisms, and $\mathcal{K}ob_{/h}(B) := \mathcal{K}om_{/h}(\mathcal{M}at(\mathcal{C}ob^3_{/\ell}(B)))$.

Theorem 2. *The assignment* $T \mapsto Kh(T) \in \mathcal{K}ob_{/h}(B)$ *is an isotopy invariant of tangles.*

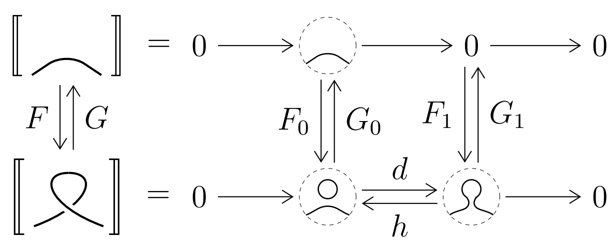
Proof. We only provide a sketch of how to prove invariance under $R1$. Let T be a tangle diagram with an $R1$ twist and T' the same tangle diagram without the mentioned $R1$ twist, so

$T =$  and $T' =$  .

Let T_0 be the diagram of T with a 0-smoothing at the crossing made by the $R1$ twist, and T_1 the diagram of T with a 1-smoothing at the same crossing, so

$T_0 =$  and $T_1 =$  .

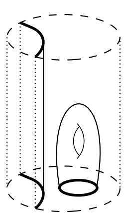
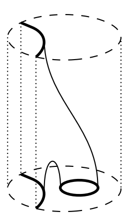
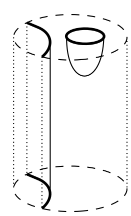
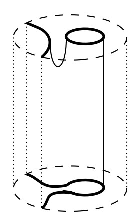
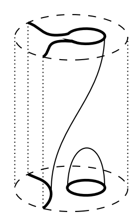
We now show there exist two morphisms (bordisms) $F : [\![T']\!] \to [\![T]\!]$ and $G : [\![T]\!] \to [\![T']\!]$ such that both compositions are null-homotopic. Diagrammatically, we show that the diagram on the right commutes.



The maps F_1 and G_1 are 0, and h is a chain homotopy as shown in the diagram of cochain complexes below.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^{i-1}(T) & \longrightarrow & C^{i}(T) & \longrightarrow & C^{i+1}(T) & \longrightarrow & \cdots \\
 & & FG \uparrow\uparrow I & \nwarrow^{h} & FG \uparrow\uparrow I & \nwarrow^{h} & FG \uparrow\uparrow I & & \\
\cdots & \longrightarrow & C^{i-1}(T) & \longrightarrow & C^{i}(T) & \longrightarrow & C^{i+1}(T) & \longrightarrow & \cdots
\end{array}$$

Define the maps

$F_0 =$  $-$  , $G_0 =$  , $d =$  , $h =$  .

The maps are given only for the region of the diagram under consideration, with the rest of the diagram having the identity map applied to it. The action of the map moves from the top of the cylinder to the bottom. To complete the proof, the identities

$$G_0F_0 = I_{T'}, \qquad G_0h = 0,$$
$$F_0G_0 + hd = I_{T_0}, \qquad dF_0 = 0$$

need to be proved, which are left as an exercise (remember to use the S, T, and $4Tu$ relations). □

We now define a category $\mathcal{C}ob^4(B)$. An object is a (based, oriented) tangle diagram, which can be thought of as a tangle in $\mathbf{R^2} \times (-\varepsilon, \varepsilon)$ whose projection to $\mathbf{R^2}$ yields the diagram. A morphism is a cobordism is $\mathbf{R^4} \cong \mathbf{R^2} \times (-\varepsilon, \varepsilon) \times \mathbf{R}$ between two such diagrams. The category $\mathcal{C}ob^4(B)_{/i}$ is defined similarly, but morphisms are taken up to isotopy.

Theorem 3. *Kh is a functor $\mathcal{C}ob^4_{/i}(B) \to \mathcal{K}ob_{/h}(B)/(\pm 1)$, where "$/(\pm 1)$" is the quotient on morphisms by ± 1.*

The first step in proving this theorem is to show that Kh is a functor $\mathcal{C}ob^4(B) \to \mathcal{K}ob_{/h}(B)$. This is not too difficult. The "elementary" cobordims in $\mathbf{R^4}$ (morphisms in the source) are given by the three Reidemeister moves, as well as the cap, cup, and saddle. One can associate a morphism in $\mathcal{K}ob$ to a Reidemeister move in the same way that one does in the proof of Theorem 2, partly sketched above. It is easy to see that the cup, cap, and saddle induce morphisms in $\mathcal{K}ob$.

The second, harder step is verifying that Kh descends to the quotient by isotopy of morphisms. Following Carter and Saito (taken as a black box here), this means verifying that certain movies (see Figures 11-13 in Bar-Natan's paper on tangles) are taken to automorphisms in $\mathcal{K}ob$ which are homotopic to ± 1. The strategy is to show that for every start/end tangle diagram T in these movies, the only automorphism of $Kh(T)$ is ± 1, up to homotopy. Showing this in turn uses a structure of "horizontally" stacking tangle diagrams side by side or, more generally, plugging tangle diagrams into other tangle diagrams.

The "horizontal" structure is the structure of a *planar algebra*, as considered by Jones (see for example `http://math.berkeley.edu/~vfr/planar.pdf`). For an algebraic topologist, the quickest way to define a planar algebra is by defining a colored operad called the *planar operad*, where the colors are the even natural numbers; a planar algebra is then just an algebra over this operad. An element in arity $(k_1, ..., k_n; k)$ is a crossingless tangle in D^2 with n holes and k_i strands touching the boundary of the i-th hole; there are k strands incident to the outer boundary circle, and each of the $n+1$ circles has a basepoint.

A tangle in $\mathcal{C}ob(B)$ can be plugged into the k_i-th hole, provided $|B| = k_i$. Thus the objects of $\mathcal{C}ob$ (taken over all B) form a planar algebra. Furthermore, it turns out that both the objects and morphisms of both $\mathcal{C}ob^4$ and $\mathcal{K}ob$ (taken over all possible B) form planar algebras. Such a category (as $\mathcal{C}ob^4$ or $\mathcal{K}ob$) with a compatible planar algebra structure is called a "canopoly" by Bar-Natan, where the terminology comes from picturing cobordisms of tangles stacked side by side as cans. A canopoly is slightly more general than a 2-category because one has not just horizontal composition (say, from an object k' to an object k) but horizontal operations (from objects $k_1, ..., k_n$ to an object k) from the planar algebra structure. In any case, the canopoly structure is useful for proving the above theorem because it allows one to prove statements for tangle diagrams by a reduction to proving statements about simpler tangle diagrams. We leave the reader to consult Bar-Natan's paper on tangles for details.

5 Topological field theory

We now provide an explicit description of the Frobenius algebra (or TQFT) coming from $H^*(S^2)$. After all, we do have to follow the functor in Theorem 3 by this TQFT to get the Khovanov homology of a link.

Recall that a 2-TFT is a symmetric monoidal functor $\mathcal{B}ord_2^{or} \to Vect$. The vector space $V = \text{span}\{v_+, v_-\} \in \text{Ob}(Vect)$, which was associated to every compact 1-manifold without boundary above is given the structure of a Frobenius algebra as below. The bordisms associated with the maps are also given.

unit	counit	multiplication	comultiplication	identity

unit	counit	multiplication	comultiplication	identity
$\begin{array}{ccc} k & \to & V \\ 1 & \mapsto & v_+ \end{array}$	$\begin{array}{ccc} V & \to & k \\ v_+ & \mapsto & 0 \\ v_- & \mapsto & 1 \end{array}$	$\begin{array}{ccc} V \otimes V & \to & V \\ v_+v_+ & \mapsto & v_+ \\ v_+v_- & \mapsto & v_- \\ v_-v_+ & \mapsto & v_- \\ v_-v_- & \mapsto & 0 \end{array}$	$\begin{array}{ccc} V & \to & V \otimes V \\ v_+ & \mapsto & v_+v_- + v_-v_+ \\ v_- & \mapsto & v_-v_- \end{array}$	$\begin{array}{ccc} V & \to & V \\ v_+ & \mapsto & v_+ \\ v_- & \mapsto & v_- \end{array}$

The S, T, and $4Tu$ relations now follow directly.

Finally, we mention that Bar-Natan uses his Theorem 3 to extend Khovanov homology to tangles, thus generalizing the Jones polynomial for tangles. This is nontrivial because the Jones polynomial for tangles takes values in the "skein module." Key ingredients for this result include defining the trace and Euler characteristic of an arbitrary pre-additive category. We again leave the reader to consult the references for details, but we point this out as an application of staying in the world of cobordisms in Theorem 3 rather than passing immediately to complexes of graded vector spaces/abelian groups.

References

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[2] Dror Bar-Natan. "On Khovanov's categorification of the Jones polynomial". In: *Algebr. Geom. Topol.* 2 (2002), 337–370 (electronic). ISSN: 1472-2747. DOI: `10.2140/agt.2002.2.337`. URL: `http://dx.doi.org/10.2140/agt.2002.2.337`.

[3] V.F.R. Jones. *Planar Algebras, I.* 1999. URL: `http://math.berkeley.edu/~vfr/planar.pdf`.

[4] Mikhail Khovanov. "A categorification of the Jones polynomial". In: *Duke Math. J.* 101.3 (2000), pp. 359–426. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-00-10131-7. URL: http://dx.doi.org/10.1215/S0012-7094-00-10131-7.