Distributional fixed points and attractors in queueing theory

Sergio López

Universidad Nacional Autónoma de México
(Joint work with Pablo Ferrari, Universidad de Buenos Aires)

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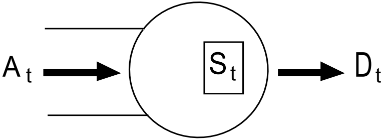
The $M \setminus M \setminus 1$ queue.



A_t arrival process.
S_t service process.

For the $M \setminus M \setminus 1$, $A_t \sim Poisson(\lambda)$, $S_t \sim Poisson(\mu)$, independent.

Stability (recurrence): $\lambda < \mu$.

D_t effective departure process.
Q_t queue length.

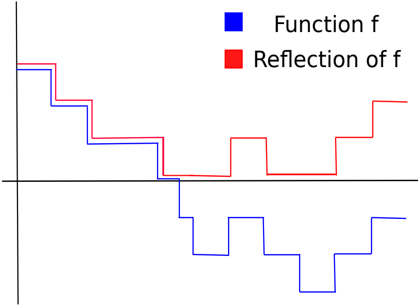
Regulator mapping

$f : [0, \infty) \to \mathbb{R}$, $f(0) \geq 0$, $g : \mathbb{R} \to \mathbb{R}$ s.t. $\lim_{t \to -\infty} g_t = \infty$ and $\lim_{t \to \infty} g_t = -\infty$.

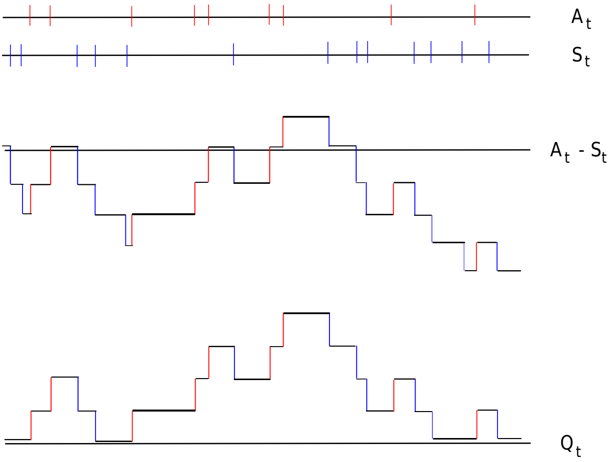
Reflective mapping:

$R(f)_t = f_t - \inf_{0 \leq s \leq t}\{f_s \wedge 0\}$,

$R(g)_t = g_t - \inf_{s \leq t}\{g_s \wedge 0\} = \sup_{s \leq t}[g_t - g_s]^+$.



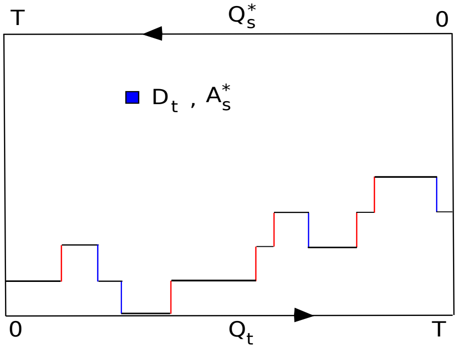
Construction using the regulator mapping



Burke's theorem

[Burke 56'] *For a stationary stable M/M/1 queue, with arrival Poisson(λ), service Poisson(μ), the departure process is a Poisson(λ) process.*

Proof of Burke's theorem:



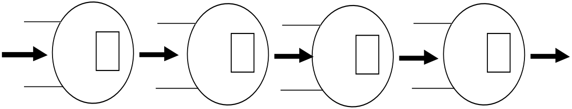
Reverse process $Q_t^* \equiv Q(T - t^-)$.

$\{A_s^* : 0 \leq s \leq T\} \stackrel{d}{=} \{A_t : 0 \leq t \leq T\}$ (reversibility)

$\{A_s^*\} \stackrel{a.s.}{=} \{D_T - D_{T-s}\}$ (identification)

$\stackrel{d}{=} \{D_t : 0 \leq t \leq T\}$ (reversibility). ■

Tandem queues



Under stationary stable *Poisson*(μ) servers and independence, if the initial arrival process has *Poisson*(λ) law, so the other arrival processes.

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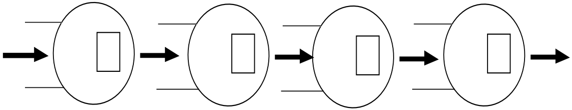

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Attractiveness

[Mountford-Prabhakar 95']

Let A an ergodic stationary point process with $\lambda < 1$ such that

$$\lim_{t\to\infty} \frac{A(t)}{t} = \lambda \qquad \mathbb{P} - a.s.$$

$\{N^n\}_{n=1}^{\infty}$ *ind. Poisson(1) process. Let*

$$A^1 = A, \qquad A^n = \mathbb{Q}(A^{n-1}, N^{n-1}) \quad \forall n \geq 2$$

where $\mathbb{Q}(A, S)$ is the departure process of the queue operator with arrival A and service S. Then A^n converges to a Poisson(λ) process.

Mountford-Prabhakar's theorem

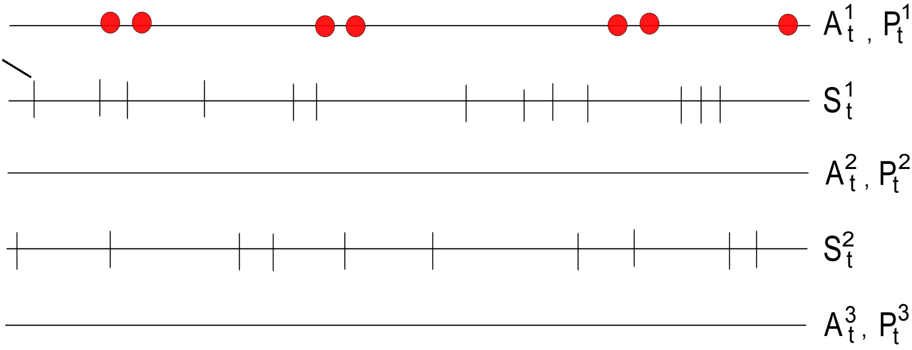
Proof's idea:

Let P an independient $Poisson(\lambda)$ process. Let it pass over the tandem queues

$$P^1 = A, \qquad P^n = \mathbb{Q}(P^{n-1}, N^{n-1}) \quad \forall n \geq 2$$

using the **same** services $\{N^n\}_{n=1}^{\infty}$.

Mountford-Prabhakar's theorem



- All clients in A^n and P^n eventually couple.
- P^n is $Poisson(\lambda)$, $\forall n$.

$\square$

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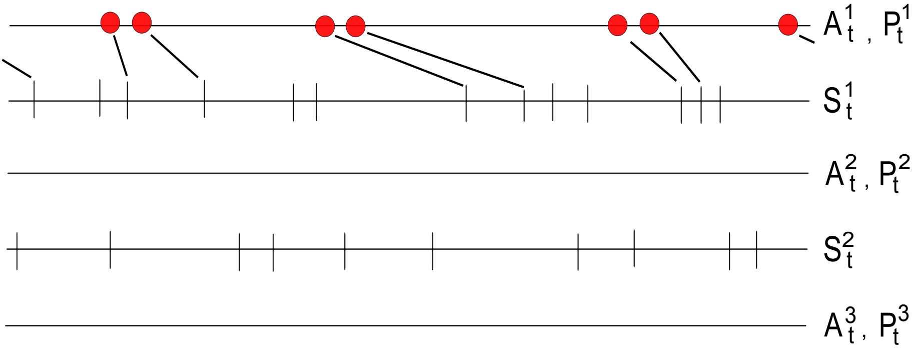

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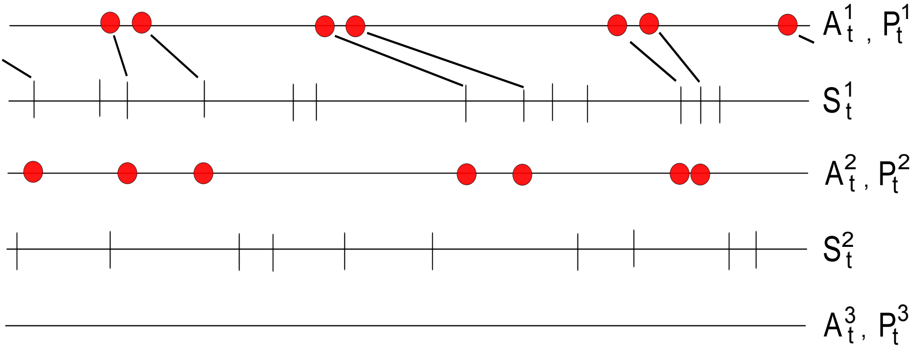

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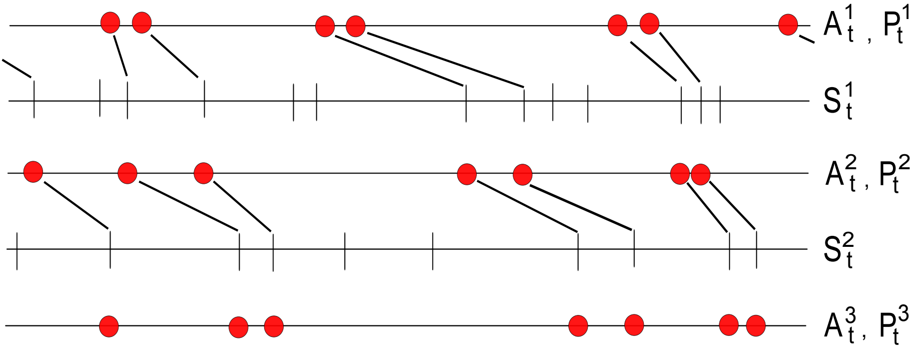

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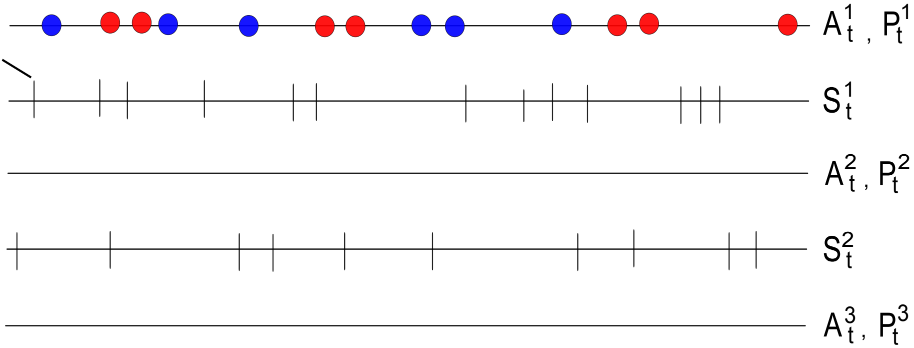

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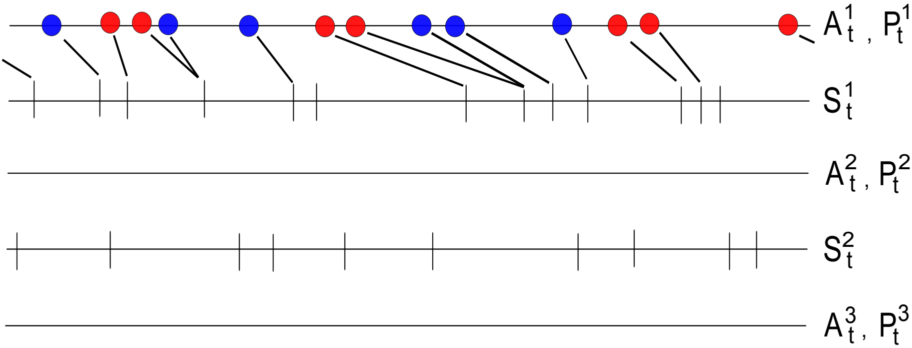

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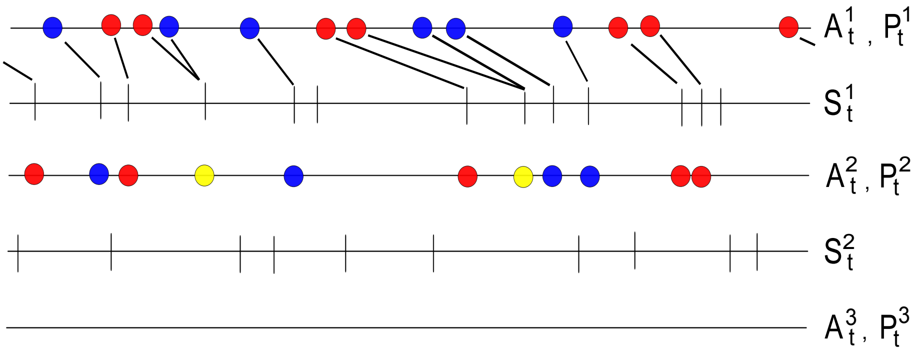

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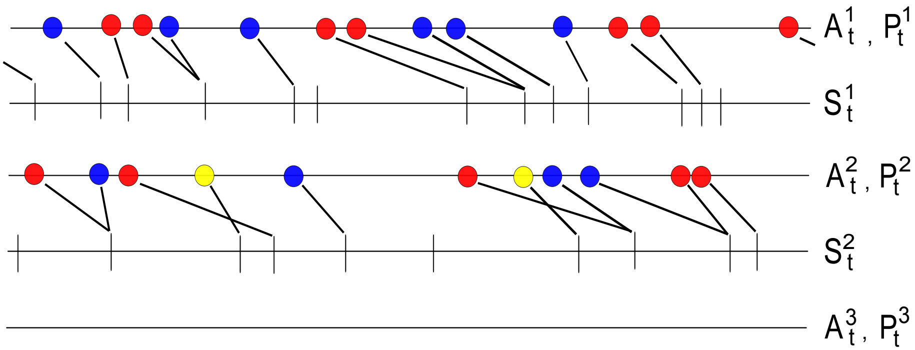

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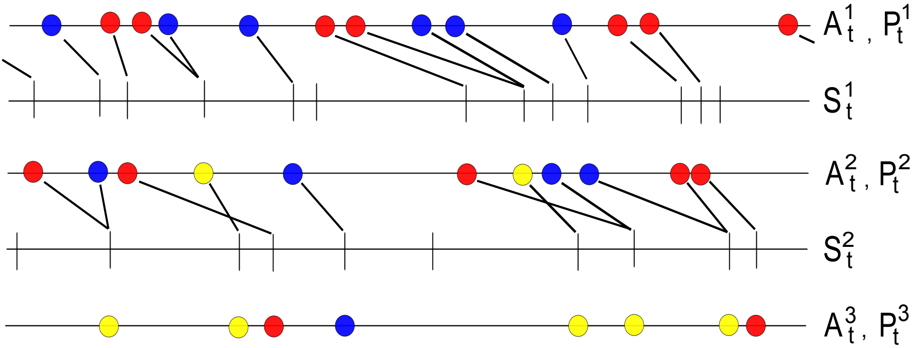

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□

Continuous valued queues

Let A_t the arrival process, S_t the service process, continuous process on $\mathbb{R}+$ (or $\mathbb{R}$).

Define $Q = R(A - S)$.

Stability: $A_t - S_t$ has negative drift (descends from infinity).

$Q_{[s,t]} = A_{[s,t]} - D_{[s,t]}$,

then $D_{[s,t]} \equiv A_{[s,t]} - Q_{[s,t]}$.

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Brownian queue

Let $B_t \perp W_t$ be standard B.M.

$A_t = B_t$, $S_t = W_t + c\,t$, $c > 0$.

$x \geq 0$.

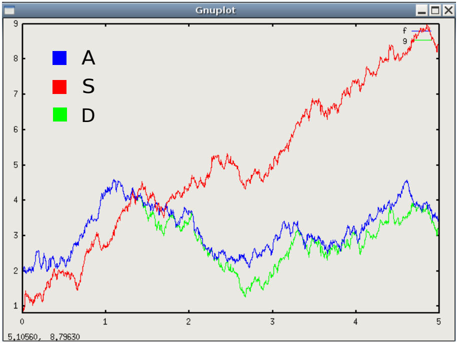
$$Q_t = R(x + A_t - S_t),$$

is a **regulated B.M.** with drift $-c < 0$.

Brownian Burke's analogue

Theorem

[Harrison 91', O'Conell-Yor 01'] Let A_t be a B.M., let $S_t + c\,t$ an ind. B.M., where $A_0 - S_0$ is $exp(c)$. Let D_t the B.M. driven by S_t with barrier A_t. Then D_t is a B.M.



Burke's analogue

Proof's idea:

- Scale a discrete valued queue, using heavy traffic limit ($\lambda_n \sim 1 - \frac{c}{\sqrt{n}}$), then use continuity of R, [Whitt 02']. Calculate limits using and not using Burke's theorem and conclude by weak limit unicity.
- Or use properties of Brownian Motion ($2M - X$ Pitman's representation theorem).

$\square$

A Brownian analogue to Mountford-Prabhakar's theorem

[Ferrari, L.] *Let A be a non-explosive continuous process. Let $\{S^n\}_{n\in\mathbb{N}}$ be B.M.'s with drift $c > 0$ and $\{E^n\}_{n\in\mathbb{N}}$ exp(c) r.v. the initial workload of each queue. Assume independence. Define*

$$A^1 = A \qquad A^n = \mathbb{Q}(A^{n-1}, S^{n-1}),$$

then A^n converges to a B.M.

M-P's analogue

Proof's idea:

Define an independent B.M. and apply the queueing operator

$$B^1 = B \qquad B^n = \mathbb{Q}(B^{n-1}, S^{n-1}),$$

with the **same** services.

[Non-increasing distance in synchrony] *Let $f, g, h : \mathbb{R}_+ \to \mathbb{R}$ be càdlàg functions with $f(0), g(0) > h(0)$. Define f^* (g^*) the function driven by h and reflected on the barrier f (g). Then,*

$$\|f^* - g^*\|_{[0,T]} \leq \|f - g\|_{[0,T]}.$$

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$$||f^* - g^*||_{[0,T]} \leq ||f - g||_{[0,T]}.$$

M-P's analogue

If $S^n \leq A^n \wedge B^n$ on $[0, T]$, the paths couple (and it persists). Define $O_n \equiv \{S^n \leq A^n \wedge B^n \text{ on } [0, T]\}$.

- Use monotonicity (derived by the lemma) and properties of B.M.
- Conclude by (a markovian version of) Borel-Cantelli lemma.

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$\square$

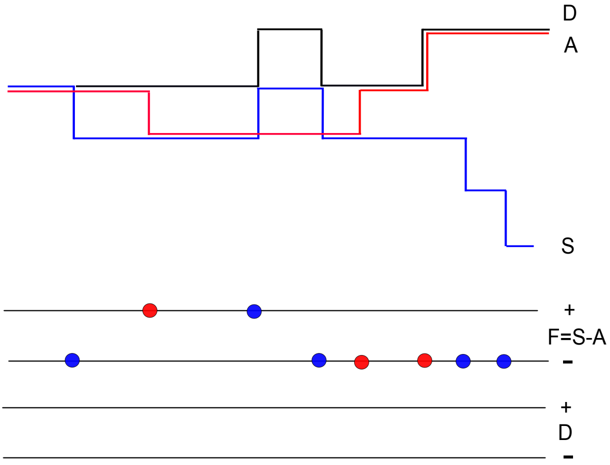
Work in progress: Towards a stationary analogue to M-P

Prove the convergence with stationary distribution in each queue instead of exp r.v.'s as initial workloads.

Obtain a stationary tandem system.

Towards a stationary analogue to M-P

[Ferrari, L.] *Graphical construction of reflections of simple random walks.*



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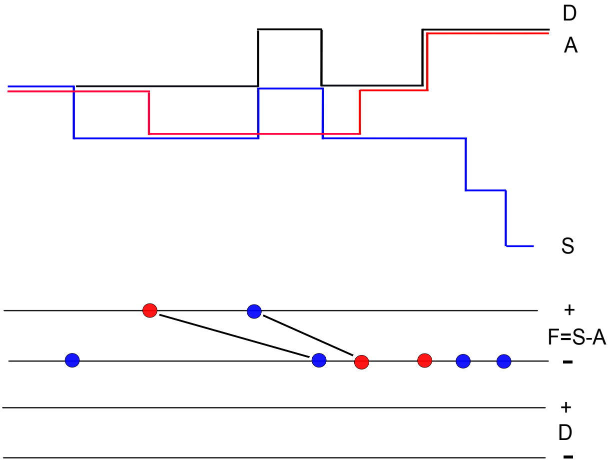

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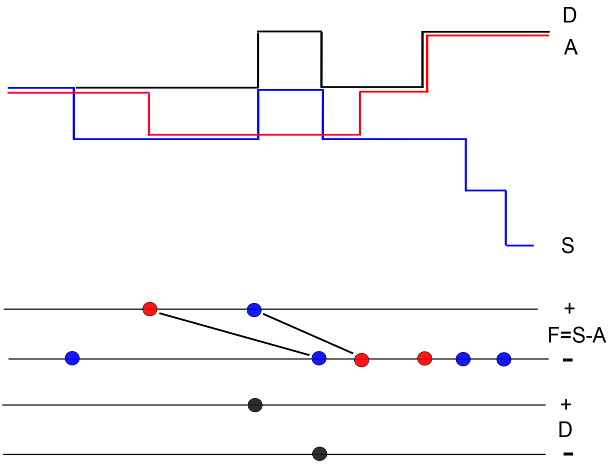

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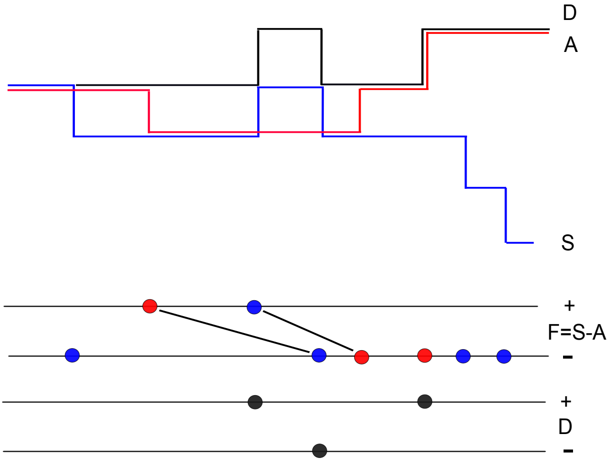

Towards a stationary analogue to M-P

[Ferrari, L.] *Non-existence of a SRW as a fixed point, i.e. reflection of a SRW over another SRW is not a SRW in any case.*

Conjecture: For SRW servers, there exists an attractive fixed point for the queueing operation which converges to BM under scaling.

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